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CYCLIC LTI SYSTEMS AND THE PARAUNITARY INTERPOLATION PROBLEM

P. P. Vaidyanathan and Ahmet Kirac

Dept. Electrical Engr., Caltech, Pasadena, CA 91125

Abstract.[1] Cyclic signal processing refers to situations where all the time indices are interpreted modulo some integer L. Since the frequency domain is a uniform discrete grid, there is more freedom in theoretical and design aspects. The basics of cyclic(L) multirate systems and filter banks have already appeared in the literature, and important differences between the cyclic and noncyclic cases are known. Since there is a strong connection between paraunitary filter banks and orthonormal wavelets, some deeper questions pertaining to cyclic(L) paraunitary matrices are addressed in this paper. It is shown that cyclic(L) paraunitary matrices do not in general have noncyclic paraunitary FIR interpolants, though IIR interpolants can always be constructed. It is shown, as a consequence, that cyclic paraunitary systems cannot in general be factored into degree one nonrecursive paraunitary building blocks. The connection to unitariness of the cyclic state space realization is also addressed.

1. INTRODUCTION

Cyclic signal processing refers to situations where all the time indices are interpreted modulo some integer L. In such cases the frequency domain is defined as a uniform discrete grid, as in L-point DFT [5],[7]. This offers more freedom in theoretical as well as design aspects. The circular convolution viewpoint has already found applications in image coding [8],[6]. Filter banks based on circular convolution have also been introduced with various different motivations, including the generation of cyclic wavelets [2],[3],[4],[10]. The fundamentals of cyclic(L) multirate systems and filter banks have also been presented recently [10], emphasizing important differences between the cyclic and noncyclic cases.

In view of the increasing interest on this topic, and since there is a strong connection between paraunitary filter banks and orthonormal wavelets [1], [9],[11] some theoretical questions pertaining to cyclic(L) paraunitary matrices were addressed in [10]. Since then a number of deeper properties have been discovered, which might be of significant interest to researchers in the filter bank and wavelet communities. The purpose of this paper is to present some of the interesting problems that connect cyclic and noncyclic paraunitary systems.

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2. CYCLIC LTI SYSTEMS

Consider two sequences $x(n)$ and $h(n)$ defined for $0 \leq n \leq L-1$, and let $y(n)$ denote their circular or cyclic convolution [7]. That is, $y(n) = \sum_{m=0}^{L-1} x(m)h(n-m)$ with all arguments interpreted modulo L. We can regard $x(n)$ and $y(n)$ as the input and output, respectively, of a linear system. With all time-arguments interpreted modulo-L, this is also a time-invariant system (i.e., a circular-shift invariant system). We say that $\{h(n)\}$ is a **cyclic LTI system**. For the purpose of interpretation one can also regard $x(n)$ to be a periodic-L input, for which the LTI system $h(n)$ yields the periodic-L output $y(n)$. This definition also extends to the multi-input multi-output case where the transfer function is a matrix sequence $\mathbf{E}(k)$, regarded as the L-point DFT

$$\mathbf{E}(k) = \sum_{n=0}^{L-1} \mathbf{e}(n) W_L^{kn}, \quad 0 \leq k \leq L-1,$$

and the input-output relation in the cyclic-time domain is $\mathbf{y}(n) = \sum_{m=0}^{L-1} \mathbf{e}(n-m)\mathbf{x}(m)$.

A **cyclic(L) paraunitary system** is defined to be a cyclic(L) LTI system whose transfer matrix $\mathbf{E}(k)$ is unitary, that is, $\mathbf{E}^\dagger(k)\mathbf{E}(k) = \mathbf{I}$ for $0 \leq k \leq L-1$. Compare this with the paraunitary property of a noncyclic system $\mathbf{E}_{non}(z)$, which is defined as $\widetilde{\mathbf{E}}_{non}(z)\mathbf{E}_{non}(z) = \mathbf{I}$. (where $\widetilde{\mathbf{E}}_{non}(z) = \mathbf{E}^\dagger_{non}(1/z^*)$). Similarly the scalar cyclic(L) system is said to be allpass if $H(k) = e^{j\theta_k}$ for $0 \leq k \leq L-1$.

The **noncyclic counterpart** of a cyclic(L) LTI system $\mathbf{E}(k)$ is defined as

$$\mathbf{E}_{nc}(z) = \sum_{n=0}^{L-1} \mathbf{e}(n) z^{-n} \tag{1}$$

This can be regarded as an interpolated version in the frequency domain, with $\mathbf{E}(k)$ representing the samples of $\mathbf{E}_{nc}(z)$ at the unit-circle points $z = W_L^{-k} = e^{j2\pi k/L}$.

Notice that the noncyclic version is only one possible interpolant. More generally, a noncyclic transfer function $\mathbf{E}_{int}(z)$ is said to be an interpolant for the cyclic(L) system $\mathbf{E}(k)$ if

$$\mathbf{E}_{int}(W_L^{-k}) = \mathbf{E}(k). \qquad (2)$$

If the interpolant is restricted to be of the form $\mathbf{E}_{int}(z) = \sum_{n=0}^{N} \mathbf{e}_{int}(n)z^{-n}$ and $N \leq L-1$ (i.e., FIR with order $\leq L-1$) then $\mathbf{E}_{nc}(z) = \mathbf{E}_{int}(z)$ because $\mathbf{e}_{int}(n)$ is then the inverse DFT of $\mathbf{E}(k)$). However, there can exist interpolants with $N \geq L$, and even IIR interpolants.

In **[10]** we showed that the noncyclic counterpart does not in general share the properties of the cyclic system. For example, $H(k)$ might be a cyclic(L) allpass filter but $H_{nc}(z)$ may not be allpass. Similarly it is possible that $\mathbf{E}(k)$ is paraunitary but not $\mathbf{E}_{nc}(z)$. This raises the following natural question: are there FIR paraunitary interpolants with higher orders, and are there IIR interpolants? In the next few sections we address various aspects of these questions, and also connect this interpolation problem to the question of factorizability of cyclic(L) paraunitary matrices.

3. PARAUNITARY INTERPOLATION

Given an arbitrary cyclic(L) paraunitary system $\mathbf{H}(k)$, that is, a sequence of $M \times M$ unitary matrices

$$\mathbf{H}(0), \mathbf{H}(1), \ldots, \mathbf{H}(L-1), \qquad (3)$$

can we always find a paraunitary interpolant $\mathbf{H}_{int}(e^{j\omega})$? This is the paraunitary interpolation problem. For the scalar case ($M = 1$) this becomes the allpass interpolation problem. For the matrix case (arbitrary M) we distinguish between the FIR case where $\mathbf{H}_{int}(z) = \sum_{n=0}^{N} \mathbf{h}(n)z^{-n}$, and the IIR case. In the IIR case we again distinguish between rational interpolants (where each element in $\mathbf{H}_{int}(z)$ is a ratio of polynomials in z^{-1}), and irrational ones.

3.1. Scalar Allpass Interpolation

Any scalar cyclic allpass filter $H(k)$ can be written in the form $H(k) = c(\sum_{n=0}^{J} b_{J-n}^* W_L^{kn})/(\sum_{n=0}^{J} b_n W_L^{kn})$ where $|c| = 1$. This shows that there exists a noncyclic allpass interpolant

$$H_{int}(z) = (c\sum_{n=0}^{J} b_{J-n}^* z^{-n})/(\sum_{n=0}^{J} b_n z^{-n})$$

where $|c| = 1$. In general, this interpolant does not have all poles inside the unit circle. For example, suppose $N = 1$ and consider the cyclic allpass filter $H(k)$ and its interpolant $H_{int}(z)$ given below:

$$H(k) = \frac{b^* + W_L^k}{1 + bW_L^k} \qquad H_{int}(z) = \frac{b^* + z^{-1}}{1 + bz^{-1}} \qquad (4)$$

Let $|b| > 1$. Then the cyclic allpass filter is still well defined (because $1 + bW_L^k \neq 0$ for any k). But the interpolant $H_{int}(z)$ represents an allpass filter with a pole outside the unit circle. This raises the following question: suppose we allow the allpass interpolant to be of higher order N (even possibly $N > L$). That is, we now have

$$H(k) = \frac{b^* + W_L^k}{1 + bW_L^k} \qquad H_{int}(z) = \frac{c\sum_{n=0}^{N} b_{N-n}^* z^{-n}}{\sum_{n=0}^{N} b_n z^{-n}} \qquad (5)$$

Can we show there exists a polynomial $\sum_{n=0}^{N} b_n z^{-n}$ of some order N with all its zeros inside the unit circle (i.e., a minimum-phase polynomial) such that $H(k) = H_{int}(W_L^{-k})$? The following example gives evidence to the contrary.

Example 1. Minimum phase interpolants. Let $B(k) = 1 + bW_L^k$ with $b > 1$ and let $B_{int}(z) = \sum_{n=0}^{N} b_n z^{-n}$ be an interpolant with real coefficients, i.e., $B(k) = B_{int}(W_L^{-k})$. We can find unlimited number of choices of N and $\{b_n\}$ satisfying this. But for even L, there does not exist even one choice such that $B_{int}(z)$ has minimum phase (i.e., all zeros in $|z| < 1$). This is proved as follows: since the coefficients $\{b_n\}$ are real, a necessary condition for $B_{int}(z)$ to have minimum phase is that $B_{int}(1)$ and $B_{int}(-1)$ have the same sign (Problem 2.13, [9]). By setting $k = 0$ in $B(k) = B_{int}(W_L^{-k})$ we find $B(0) = B_{int}(1)$. Similarly $B(L/2) = B_{int}(-1)$. Using $b > 1$ we therefore conclude

$$B_{int}(1) = 1 + b > 0, \quad B_{int}(-1) = 1 - b < 0.$$

Thus $B_{int}(1)$ and $B_{int}(-1)$ could never have the same sign no matter how we choose N and $\{b_n\}$. Summarizing, $B(k)$ does not have a minimum phase interpolant $B_{int}(z)$. However, it is still conceivable that the ratio $\widetilde{B}_{int}(z)/B_{int}(z)$ has some cancellations, thereby resulting in a stable allpass interpolant. Moreover, we have not considered the possibility of a complex-coefficient interpolant (which is conceivable even for real b).

3.2. FIR Paraunitary Interpolation

Given the unitary sequence (3), can we always find a paraunitary interpolant restricted to be FIR, i.e., of the form $\mathbf{H}_{int}(z) = \sum_{n=0}^{N} \mathbf{h}(n)z^{-n}$? For the scalar case ($M = 1$) the answer is evidently no, because the interpolant has to be an FIR allpass function (which cannot be more general than a mere delay). So we assume $M \neq 1$. If we make the restriction $N < L$ the coefficients $\mathbf{h}(n)$ are simply the inverse DFT coefficients of $\mathbf{H}_{int}(W_L^{-k})$ and the interpolant is the noncyclic counterpart defined in Sec. 2. This may not be paraunitary as seen from the examples of **[10]**. More generally, suppose we allow N to be arbitrarily large but finite. Does this allow us to always find a paraunitary interpolant? In general the answer is still no, as we shall demonstrate. For this we first review a well known result for noncyclic 2×2 causal FIR paraunitary matrices [9]:

Theorem 1. Let $\mathbf{H}_{non}(z) = \sum_{n=0}^{N} \mathbf{h}(n)z^{-n}$ be a 2×2 causal FIR paraunitary system. Then it has the general form

$$\mathbf{H}_{non}(z) = \begin{bmatrix} H_0(z) & e^{j\theta}z^{-n_0}\widetilde{H}_1(z) \\ H_1(z) & -e^{j\theta}z^{-n_0}\widetilde{H}_0(z) \end{bmatrix} \qquad (6)$$

where $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_1(z)H_1(z) = 1$ (power complementary property), θ is real, and n_0 is any integer large enough to ensure causality. ◇

Consider now the cyclic(L) example

$$\mathbf{H}(k) = \begin{bmatrix} e^{j\alpha(k)} & 0 \\ 0 & e^{j\beta(k)} \end{bmatrix} \qquad (7)$$

which is evidently paraunitary for any arbitrary (real-valued) $\alpha(k)$ and $\beta(k)$. Suppose there exists a noncyclic causal FIR paraunitary interpolant $\mathbf{H}_{int}(z)$. If this interpolant is diagonal, then $H_0(e^{j\omega}) = ce^{-j\omega N}$ and $\alpha(k)$ has to be of the form $-2\pi kN/L$ plus a constant. Thus for arbitrary $\alpha(k)$ and $\beta(k)$, there is no diagonal FIR paraunitary interpolant $\mathbf{H}_{int}(z)$. How about a nondiagonal interpolant? Since it would have the form (6), $\alpha(k)$ and $\beta(k)$ cannot have arbitrary combinations of values. Only combinations satisfying

$$\beta(k) = -\alpha(k) - 2\pi kn_0/L + \delta$$

for some δ are allowed. This shows that *cyclic paraunitary systems do not in general have noncyclic FIR paraunitary interpolants.* Whenever such an interpolant does exist, it can be factorized into the form [9]

$$\mathbf{H}_{int}(z) = \mathbf{H}_{int}(1)\prod_{i=1}^{J}\left(\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + z^{-1}\mathbf{u}_i\mathbf{u}_i^\dagger\right) \qquad (8)$$

where $\mathbf{u}_i$ are unit norm vectors. By replacing z^{-1} with W_L^k we obtain

$$\mathbf{H}(k) = \mathbf{H}(0)\prod_{i=1}^{J}\left(\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + W_L^k\mathbf{u}_i\mathbf{u}_i^\dagger\right) \qquad (9)$$

which is a factorization of the cyclic paraunitary system $\mathbf{H}(k)$ in terms of the building blocks $\left(\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + W_L^k\mathbf{u}_i\mathbf{u}_i^\dagger\right)$. Thus, whenever a noncyclic FIR paraunitary interpolant exists, the cyclic system can be factored as above. Conversely, if the cyclic system can be factored as in (9), then replacing W_L^k with z^{-1} we obtain a FIR paraunitary interpolant. Summarizing we have proved:

Theorem 2. FIR interpolants. Let $\mathbf{H}(k)$ be a cyclic(L) paraunitary system, (i.e., $\mathbf{H}(k)$ unitary for $0 \le k \le L-1$). Then it has a noncyclic causal FIR paraunitary interpolant $\mathbf{H}_{int}(z)$ if and only if $\mathbf{H}(k)$ can be factorized in the form (9) where $\mathbf{u}_i$ are unit-norm vectors. ◇

4. IIR PARAUNITARY INTERPOLATION

If we do not restrict the interpolant to be FIR, then we can always find a paraunitary interpolant for the unitary sequence (3). For this we simply define,

$$\mathbf{H}_{int}(e^{j\omega}) = \mathbf{H}(k), \qquad \frac{2\pi k}{L} \le \omega < \frac{2\pi(k+1)}{L} \qquad (10)$$

for $0 \le k \le L-1$. Then the sample values $\mathbf{H}_{int}(W_L^{-k})$ are evidently equal to $\mathbf{H}(k)$. The interpolant $\mathbf{H}_{int}(e^{j\omega})$ is a piecewise constant, and has discontinuities at the frequencies $2\pi k/L$. It is therefore not a rational function in $e^{j\omega}$ (i.e., the elements in the matrix are not ratios of polynomials in z). The following theorem asserts that we can *always construct rational solutions* in the IIR case. Stability of the interpolant, however, is not asserted.

Theorem 3. IIR interpolants. Let $\mathbf{H}(0), \mathbf{H}(1), \ldots \mathbf{H}(L-1)$ be a sequence of $M \times M$ unitary matrices. Then there exists a causal system with rational transfer matrix $\mathbf{H}_{int}(z)$ such that $\mathbf{H}_{int}(W_L^{-k}) = \mathbf{H}(k)$. ◇

Proof. The crucial building block is the matrix

$$\mathbf{U}_k(z) \triangleq \mathbf{I} - \mathbf{u}\mathbf{u}^\dagger + z^{-1}F_k(z)\mathbf{u}\mathbf{u}^\dagger$$

where $F_k(z)$ is a rational allpass filter and $\mathbf{u}$ is a unit-norm vector. We can verify that $\mathbf{U}_k(z)$ is paraunitary. Suppose the allpass filter $F_k(z)$ is chosen such that

$$F_k(W_L^{-m}) = \begin{cases} W_L^{-m} & m \ne k \\ -W_L^{-m} & m = k \end{cases}$$

We can regard $F_k(z)$ as a rational allpass interpolant (Sec. 3.1) with samples at $z = W_L^{-m}$ as specified above. With this choice of $F_k(z)$, the matrix $\mathbf{U}_k(z)$, sampled at $z = W_L^{-m}$, yields

$$\mathbf{U}_k(W_L^{-m}) = \begin{cases} \mathbf{I} & \text{for all } m \ne k \\ \mathbf{I} - 2\mathbf{u}\mathbf{u}^\dagger & \text{for } m = k \end{cases}$$

Now any $M \times M$ unitary matrix can be expressed as a product of $M-1$ matrices of the form $\mathbf{I} - 2\mathbf{u}\mathbf{u}^\dagger$. More precisely [9] each matrix $\mathbf{H}(k)$ in the given unitary sequence can be expressed as

$$\mathbf{H}(k) = \left(\prod_{i=1}^{M-1}\left(\mathbf{I} - 2\mathbf{u}_{i,k}\mathbf{u}_{i,k}^\dagger\right)\right)\mathbf{\Lambda}(k)$$

where $\mathbf{u}_{i,k}$ are unit-norm vectors and $\mathbf{\Lambda}(k)$ is diagonal with nth diagonal element $e^{j\theta_{k,n}}$. We can find a rational allpass filter $F_n(z)$ such that

$$F_n(W_L^{-m}) = \begin{cases} 1 & m \ne k \\ e^{j\theta_{k,n}} & m = k \end{cases}$$

Then the diagonal matrix $\mathbf{D}_k(z)$ with diagonal elements $F_n(z)$ has the unit-circle samples

$$\mathbf{D}_k(W_L^{-m}) = \begin{cases} \mathbf{I} & m \ne k \\ \mathbf{\Lambda}(k) & m = k. \end{cases}$$

By multiplying matrices of the form $\mathbf{U}_k(z)$ and $\mathbf{D}_k(z)$, we can define a noncyclic paraunitary system $\mathbf{G}_k(z)$ such that

$$\mathbf{G}_k(W_L^{-m}) = \begin{cases} \mathbf{I} & m \neq k \\ \mathbf{H}(k) & m = k \end{cases}$$

The product $\mathbf{H}_{int}(z) = \mathbf{G}_0(z)\mathbf{G}_1(z)\ldots\mathbf{G}_{L-1}(z)$ then represents a rational IIR paraunitary interpolant for the given matrix sequence $\{\mathbf{H}(k)\}$. $\nabla\nabla\nabla$

5. UNITARINESS OF REALIZATION MATRIX

Suppose we are given an implementation for a cyclic transfer matrix $\mathbf{E}(k)$. This implementation has a state space description of the form [10] $\mathbf{v}(n+1) = \mathbf{A}\mathbf{v}(n) + \mathbf{B}\mathbf{x}(n)$ and $\mathbf{y}(n) = \mathbf{C}\mathbf{v}(n) + \mathbf{D}\mathbf{x}(n)$. The realization matrix for the implementation is defined as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{11}$$

The following result connects the cyclic-paraunitary property to unitariness of the realization matrix.

Lemma 1. If the realization matrix is unitary, then the cyclic system $\mathbf{E}(k)$ is paraunitary.

Proof. Rewrite the state equations as

$$\begin{bmatrix} \mathbf{v}(n+1) \\ \mathbf{y}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{v}(n) \\ \mathbf{x}(n) \end{bmatrix}$$

Unitariness of the realization matrix implies $\|\mathbf{v}(n+1)\|^2 + \|\mathbf{y}(n)\|^2 = \|\mathbf{v}(n)\|^2 + \|\mathbf{x}(n)\|^2$ where $\|\mathbf{v}\|^2$ denotes $\mathbf{v}^\dagger\mathbf{v}$. If we write the preceding equation for $0 \leq n \leq L-1$ and add them up, we obtain

$$\sum_{n=0}^{L-1} \mathbf{y}^\dagger(n)\mathbf{y}(n) = \sum_{n=0}^{L-1} \mathbf{x}^\dagger(n)\mathbf{x}(n)$$

by using the fact that $\mathbf{v}(n+L) = \mathbf{v}(n)$. With $\mathbf{X}(k) = \sum_{n=0}^{L-1} \mathbf{x}(n)W_L^{nk}$ and $\mathbf{Y}(k) = \sum_{n=0}^{L-1} \mathbf{y}(n)W_L^{nk}$, we then obtain (using Parseval's relation) $\sum_{k=0}^{L-1} \mathbf{Y}^\dagger(k)\mathbf{Y}(k) = \sum_{k=0}^{L-1} \mathbf{X}^\dagger(k)\mathbf{X}(k)$, that is,

$$\sum_{k=0}^{L-1} \mathbf{X}^\dagger(k)\mathbf{E}^\dagger(k)\mathbf{E}(k)\mathbf{X}(k) = \sum_{k=0}^{L-1} \mathbf{X}^\dagger(k)\mathbf{X}(k)$$

This should hold for all sequences $\{\mathbf{X}(k)\}$, which implies that $\mathbf{X}^\dagger(k)\mathbf{E}^\dagger(k)\mathbf{E}(k)\mathbf{X}(k) = \mathbf{X}^\dagger(k)\mathbf{X}(k)$ for any $\mathbf{X}(k)$, proving $\mathbf{E}^\dagger(k)\mathbf{E}(k) = \mathbf{I}$ indeed. $\nabla\nabla\nabla$

This result is analogous to a result in the noncyclic case [9]. However, unlike in the noncyclic case, we do not have the converse result. That is, even if $\mathbf{E}(k)$ is paraunitary, there may not exist a *minimal nonrecursive structure* with unitary system matrix. When such a structure does exist, the FIR interpolant $\mathbf{E}_{int}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, obtained by replacing W_L^k with z^{-1} in the structure, would be paraunitary (because the converse part holds in the noncyclic case [9]). Since FIR paraunitary interpolants do not always exist (Theorem 2), the point is proved.

6. CONCLUDING REMARKS

It is well known that noncyclic FIR paraunitary systems can be factored [9] in terms of the building blocks $\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + z^{-1}\mathbf{u}_i\mathbf{u}_i^\dagger$, where $\mathbf{u}_i^\dagger\mathbf{u}_i = 1$. But in the cyclic case, factorization in terms of $(\mathbf{I} - \mathbf{u}_i\mathbf{u}_i^\dagger + W_L^k\mathbf{u}_i\mathbf{u}_i^\dagger)$ is not always possible. In fact Theorem 2 shows that such factorization is possible if and only if there exists an FIR paraunitary interpolant. However, the fact that there always exists a rational IIR interpolant (Theorem 3) means, in particular, that we can obtain a factorization of the cyclic system $\mathbf{H}(k)$ in terms of slightly modified building blocks. These have the form $\mathbf{U}_k(z)$ and $\mathbf{D}_k(z)$ given in the proof of Theorem 3, with z replaced by W_L^{-k} everywhere.

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